

ON THE RANK OF A PRODUCT OF MANIFOLDS

FRANCISCO-JAVIER TURIEL AND ARTHUR G. WASSERMAN

ABSTRACT. This note gives an example of closed smooth manifolds M and N for which the rank of $M \times N$ is strictly greater than $\text{rank}M + \text{rank}N$.

Milnor defined the *rank* of a smooth manifold M as the maximal number of commuting vector fields on M that are linearly independent at each point.

One of the questions raised by Milnor at the Seattle Topology Conference of 1963, and echoed by Novikov [2], was

$$\text{is } \text{rank}(M \times N) = \text{rank}(M) + \text{rank}(N)$$

whenever M and N are smooth closed manifolds?

In this note we give a negative answer to this question.

We need a simple result about mapping tori.

Let $f: X \rightarrow X$ be a diffeomorphism of a manifold X and let

$$M(f) = \frac{I \times X}{(0, x) \sim (1, f(x))}$$

be the mapping torus of f where $I = [0, 1]$.

Equivalently, $M(f) = \frac{\mathbb{R} \times X}{\mathbb{Z}}$ where the action of \mathbb{Z} on $\mathbb{R} \times X$ is given by $\alpha(k)(t, x) = (t + k, f^k(x))$. $M(f)$ is a fibre bundle over S^1 with fiber X . We note that $\pi_1(M(f)) = \pi_1(X) *_f \mathbb{Z}$ where $*$ denotes the semi-direct product and $f_*: \pi_1(X) \rightarrow \pi_1(X)$.

Proposition 1. *Consider two periodic diffeomorphisms $f: X \rightarrow X$ and $g: Y \rightarrow Y$ with periods m and n respectively. Assume m and n are relatively prime, i.e., there are integers c, d such that $mc + nd = 1$.*

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Then $M(f) \times M(g)$ is diffeomorphic to $M(h)$ where $h: S^1 \times X \times Y \rightarrow S^1 \times X \times Y$ is defined by $h(\theta, x, y) = (\theta, f^{-d}(x), g^c(y))$. Moreover $h^{m-n} = (id, f, g)$.

Proof. $M(f) \times M(g)$ can be identified with the quotient of $\mathbb{R}^2 \times X \times Y$ under the action of \mathbb{Z}^2 given by $\beta(z)(u, x, y) = (u + z, f^{z_1}(x), g^{z_2}(y))$, where $z = (z_1, z_2) \in \mathbb{Z}^2$, $u = (u_1, u_2) \in \mathbb{R}^2$ and $(x, y) \in X \times Y$.

Set $\lambda = (m, n)$ and $\mu = (-d, c)$. Since $mc + nd = 1$, $\mathcal{B} = \{\lambda, \mu\}$ is at the same time a basis of \mathbb{Z}^2 as a \mathbb{Z} -module and a basis of \mathbb{R}^2 as a vector space. On the other hand

$$\beta(\lambda)(u, x, y) = (u + \lambda, x, y) \quad \text{and} \quad \beta(\mu)(u, x, y) = (u + \mu, f^{-d}(x), g^c(y)).$$

Therefore the action β referred to the new basis \mathcal{B} of \mathbb{Z}^2 and \mathbb{R}^2 is written now:

$$\beta(k, r)(a, b, x, y) = (a + k, b + r, \varphi^r(x), \gamma^r(y))$$

where $\varphi = f^{-d}$ and $\gamma = g^c$.

As the action of the first factor of \mathbb{Z}^2 on $X \times Y$ is trivial, identifying S^1 with $\frac{\mathbb{R}}{\mathbb{Z}}$ shows that $M(f) \times M(g)$ is diffeomorphic to $M(h)$.

Finally from $(-n)(-d) = 1 - cm$ and $cm = 1 - dn$ follows that $h^{m-n} = (id, f, g)$. \square

On the other hand:

Lemma 1. *Let $f: N \rightarrow N$ be a diffeomorphism and let X_1, \dots, X_k be a family of commuting vector fields on N that are linearly independent everywhere. Assume $f_*X_i = \sum_{j=1}^k a_{ij}X_j$, $i = 1, \dots, k$, where the matrix $(a_{ij}) \in GL(k, \mathbb{R})$. Then $\text{rank}(M(f)) \geq k$.*

Proof. It suffices to construct k commuting vector fields $\tilde{X}_1, \dots, \tilde{X}_k$ on $I \times N$ that are linearly independent at each point and such that every $\tilde{X}_i(t, x)$ equals $X_i(x)$ if t is close to zero and $f_*X_i(x)$ when t is close to 1 (X_1, \dots, X_k are considered vector fields on $I \times N$ in the obvious way).

If $|a_{ij}| > 0$ consider an interval $[a, b] \subset (0, 1)$ and a (differentiable) map $(\varphi_{ij}): I \rightarrow GL(k, \mathbb{R})$ such that $\varphi_{ij}([0, a]) = \delta_{ij}$ and $\varphi_{ij}([b, 1]) = a_{ij}$, and set $\tilde{X}_i(t, x) = \sum_{j=1}^k \varphi_{ij}(t) X_j(x)$.

When $|a_{ij}| < 0$ first take an interval $[c, d] \subset (0, 1/2)$ and a function $\rho: [0, 1/2] \rightarrow \mathbb{R}$ such that $\rho([0, c]) = 1$, $\rho([d, 1/2]) = -1$, and on $[0, 1/2] \times N$ set $\tilde{X}_1(t, x) = \rho(t)X_1(x) + (1 - \rho^2(t))\frac{\partial}{\partial t}$ and $\tilde{X}_i(t, x) = X_i(x)$, $i = 2, \dots, k$.

The matrix of coordinates of f_*X_1, \dots, f_*X_k with respect to the basis $\{-X_1, X_2, \dots, X_k\}$ has positive determinant, so by doing as before we can extend $\tilde{X}_1, \dots, \tilde{X}_k$ to $[1/2, 1] \times N$ by means of an interval $[a, b] \subset (1/2, 1)$ and a suitable map $(\varphi_{ij}): [1/2, 1] \rightarrow GL(k, \mathbb{R})$. \square

Proposition 1 and Lemma 1 quickly yield a counterexample.

Assume X is a torus $\mathbb{T}^k = \frac{\mathbb{R}^k}{\mathbb{Z}^k}$ and f is the map induced by a nontrivial element of $GL(k, \mathbb{Z})$. Then by the above lemma applied to $\frac{\partial}{\partial \theta_j}$, $j = 1, \dots, k$, $rank(M(f)) \geq k$. But $M(f)$ has non-abelian fundamental group, so it is not a torus and $rank(M(f)) = k$. (If M is a closed connected n -manifold of rank n then M is diffeomorphic to the n -torus.)

For the same reason if $Y = \mathbb{T}^r$ and g is induced by a nontrivial element of $GL(r, \mathbb{Z})$ then $rank(M(g)) = r$.

If f and g are periodic with relatively prime periods m and n respectively then by Proposition 1 $M(f) \times M(g) = M(h)$ where $h: \mathbb{T}^{k+r+1} \rightarrow \mathbb{T}^{k+r+1}$ is induced by a nontrivial element of $GL(k+r+1, \mathbb{Z})$. Moreover $rank(M(h)) = k + r + 1$. Therefore:

$$rank(M(f) \times M(g)) > rank(M(f)) + rank(M(g)).$$

For instance, set $k = r = 2$ and consider f, g induced by the elements in $SL(2, \mathbb{Z}) \subset GL(2, \mathbb{Z})$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

respectively, so $M(f)$ and $M(g)$ are orientable. Then the period of f is 2 and that of g equals 3.

An even simpler but non-orientable counterexample can be constructed as follows. Take r and g as before, $k = 1$ and f induced by (-1) . Then $M(f)$ is the Klein bottle which has rank 1 and $M(g)$ has rank 2; however, $M(f) \times M(g)$ is diffeomorphic to $M(h)$ and hence has rank 4.

Remark 1. The *file* of a manifold M was defined by Rosenberg [3] to be the largest integer k such that \mathbb{R}^k acts locally free on M . When M is closed *file*(M) equals *rank*(M) but *file*($\mathbb{R} \times S^2$) = 1, [3], while *rank*($\mathbb{R} \times S^2$) = 3.

The analog of Milnor's question for the file of a product of non-compact manifolds also fails. Indeed, let \mathbb{R}_e^4 be any exotic \mathbb{R}^4 . Then *file*(\mathbb{R}_e^4) \leq 3 otherwise $\mathbb{R}_e^4 = \mathbb{R}^4$. But $\mathbb{R}_e^4 \times \mathbb{R} = \mathbb{R}^5$ because there is no exotic \mathbb{R}^5 , so *file*($\mathbb{R}_e^4 \times \mathbb{R}$) = 5 $>$ *file*(\mathbb{R}_e^4) + *file*(\mathbb{R}).

Orientable closed connected n -manifolds of rank $n - 1$ are completely described in [4, 1, 5].

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(F.J. Turiel) GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS, CAMPUS DE
TEATINOS, S/N, 29071-MÁLAGA, SPAIN

E-mail address: `turiel@uma.es`

(A. G. Wasserman) UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109-1003,
USA

E-mail address: `awass@umich.edu`